Ayres Freitas http://www.pitt.edu/~afreitas/phy1370.html

Homework for week of 09/08/2008

Deadline 09/15/2008

Problem 1: *(5 points)*

Consider the wave function

$$\Psi(x,t) = A\,e^{-\lambda|x|}\,e^{-i\omega t}, \tag{1}$$

where A, λ, and ω are positive real constants.

a) Normalize Ψ.

b) Determine $\langle x\rangle$ and $\langle x^2\rangle$.

c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $\langle x\rangle \pm \sigma_x$, to illustrate the sense in which σ_x represents the spread in x. What is the probability that the particle would be found outside this range?

d) Calculate the expectation values of p and p^2. For $\langle p^2\rangle$ use the integration-by-parts relation

$$\int_{-\infty}^{\infty} dx\,\Psi^* \frac{\partial^2\Psi}{\partial x^2} = -\int_{-\infty}^{\infty} dx \left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{\partial\Psi}{\partial x}\right). \tag{2}$$

e) Check whether σ_x and σ_p are consistent with the uncertainty principle.

Problem 2: Show that *(2 points)*

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle. \tag{3}$$

This relation is an example of *Ehrenfest's theorem*, which states that expectation values obey classical laws.

Problem 3: *(1 point)*

Show that if $\psi(x)$ has mean momentum $\langle p\rangle$, $e^{ip_0x/\hbar}\psi(x)$ has mean momentum $\langle p\rangle + p_0$.

Problem 4: *(1 point)*

Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle p\rangle = 0$.
Hint: Use integration by parts.

Problem 5: *(2 points)*

Suppose you add a constant (independent of x and t) V_0 to the potential energy. In *classical* mechanics this does not change anything, but what about *quantum* mechanics? Show that the wave function picks up a time-dependent phase factor $e^{-iV_0 t/\hbar}$. What effect does this have on the expectation value of a dynamical variable?

Problem 6: *(2 points)*

Let $P_{ab}(t)$ be the probability of finding a particle in the range $a < x < b$, at time t.

a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t), \tag{4}$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right). \tag{5}$$

What are the units of $J(x,t)$?
Comment: J is called *probability current*, because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing then more probability is flowing into the region at one end than flows out at the other end.

b) Find the probability current for the wave function in Problem 1.

Problem 7: *(3 points)*

Prove the following three theorems:

a) For a normalizable solution the separation constant E must be real.
Hint: Write E in $\Psi(x,t) = \psi(x)e^{-tEt/\hbar}$ as $E_0 + i\Gamma$ (with E_0 and Γ real) and check the time dependence of the normalization.

b) The time-independent wave function $\psi(x)$ can always be taken to be real. Of course there are also complex solutions to the time-independent Schrödinger equation, but they can always be expressed as a linear combination of real solutions with the same energy.
Hint: First show that if $\psi(x)$ is a solution for a given E, so too is ψ^*, as well as $(\psi+\psi^*)$ and $i(\psi-\psi^*)$.

c) If $V(x)$ is an *even* function [i.e. $V(x) = V(-x)$] then $\psi(x)$ can always be taken to be either even or odd.
Hint: First show that if $\psi(x)$ is a solution for a given E, so too is $\psi(-x)$, and hence also $\psi(x) \pm \psi(-x)$.