# Game Theory Principles IX

Andreas Blume
Department of Economics
University of Pittsburgh
Pittsburgh, PA 15260

Game theory in physics and other ways of changing the game: Quantum games, pre-play communication, correlation.

Papers to read:

1. Steven E. Landsburg, "Nash Equilibria in Quantum Games"
2. David K. Levine, "Quantum Games have no News for Economists"

Expanding strategy spaces:

1. Pure strategies
2. Mixed strategies
3. Strategies that condition on prior events (e.g. in a repeated game, or following pre-play communication)
4. Strategies that condition on external signals (e.g. the time of day, the phases of the moon, sunspots)
5. Quantum strategies (appeal to the possibility of communication through quantum channels, i.e. through interaction with small particles)

Rather than just allowing for the choice of pure strategies (e.g. "cooperate" or "defect") or mixed strategies (a probability distribution over "cooperate" and "defect"), quantum strategies allow for **superpositions** (states that allow for both "cooperate" and "defect" until a measurement is made).

Why might we be interested in quantum games:

1. Quantum communication and quantum computing might become reality in the not so distant future

2. Quantum strategies permit cooperation in the one-shot prisoners' dilemma

Why might we not be interested in quantum games:

1. The transformation of a classical game into a quantum game is just one more way of studying a classical game that is embedded into a larger game.
2. Explicitly adding strategies to a classical game (that allow for commitment for example) achieves a similar effect.

**Quantum Games:**

1. Take a classical game with pure strategies $C$ and $D$.
2. Imagine the game is played by players returning pennies to a referee, who then computes payoffs. When returning the penny to the referee the penny can be either unchanged, $C$, or flipped, $D$.

3. A quantum penny, when returned to the referee, need not be in either of the two states. It can be in a state where, when it is observed by the referee there is a one-third chance of it being unchanged and a two-thirds chance of it begin flipped.

   This does not offer anything essentially new, as it is only an alternative way of implementing a mixed strategy.

Suppose one player uses a mixed-strategy with probability $p$ of $C$ and the other player uses a mixed strategy that assigns probability $q$ to $C$. If players mix independently, then the joint distribution over outcomes equals

| | C | D |
|---|---|---|
| C | $pq$ | $p(1-q)$ |
| D | $(1-p)q$ | $(1-p)\times$ $(1-q)$ |

Game IX-1

Note that with independent mixing not all probability distributions over the four outcomes are possible. For example, the following distribution cannot be achieved with independent mixing:

| | C | D |
|---|---|---|
| C | $\frac{1}{3}$ | $\frac{1}{3}$ |
| D | 0 | $\frac{1}{3}$ |

Game IX-2

4. Quantum pennies can do more than simply implement mixed strategies. With a pair of **entangled** quantum pennies it is possible to induce any probability distribution over outcomes. This in itself also does not add anything essentially new, because it is simply an alternative way of implementing correlated strategies that induce any arbitrary (correlated) distribution over outcomes.

5. Quantum pennies can do more than simply implement **correlated strategies**. Even for fixed behavior of player 2, player 1 can achieve any joined distribution over outcomes.

6. The quantum game $G^Q$ corresponding to the classical game $G$ can be thought of as a game in which the players communicate their strategies to a referee via entangled pennies.
7. A player's strategy set in the quantum game consists of all physical manipulations of his penny.

## Some obscure technical remarks

1. The manipulations of the “pennies” mathematically can be identified with “unit quaternions”
2. Hence, strategies can be identified with unit quaternions.
3. A quaternion looks like this

$$p_1 + p_2 i + p_3 j + p_4 k$$

4. Multiplication and addition of quaternions work as you would expect, with the additional rules:

$$i^2 = j^2 = k^2 = -1,$$

and

$$ij = -ji = k, \quad ik = -ki = j, \quad jk = -kj = i.$$

5. A quaternion is a unit quaternion if

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$$

6. For a quaternion $\mathbf{p} = p_1 + p_2 i + p_3 j + p_4 k$, define $\pi_i(\mathbf{p}) := p_i$
7. Correspondingly, we can write the product $\mathbf{pq}$ of two quaternions as

$$\mathbf{pq} = \pi_1(\mathbf{pq}) + \pi_2(\mathbf{pq})i + \pi_3(\mathbf{pq})j + \pi_4(\mathbf{pq})k$$

## Back to games

1. Suppose one player's strategy is the quaternion $\mathbf{p}$ and the other player's strategy is the quaternion $\mathbf{q}$. Then we get the following probability distribution over outcomes:

$$
\begin{aligned}
\text{Prob(unchanged, unchanged)} &= \pi_1(\mathbf{pq})^2 \\
\text{Prob(unchanged, flipped)} &= \pi_2(\mathbf{pq})^2 \\
\text{Prob(flipped,unchanged)} &= \pi_3(\mathbf{pq})^2 \\
\text{Prob(flipped, flipped)} &= \pi_4(\mathbf{pq})^2
\end{aligned}
$$

2. Consider a classical game $G$ with payoff matrix

| | C | D |
|---|---|---|
| C | $X_1, Y_1$ | $X_2, Y_2$ |
| D | $X_3, Y_3$ | $X_4, Y_4$ |

GAME IX-3

3. Then the payoffs in the corresponding quantum game are given by

$$\Pi_1(\mathbf{p},\mathbf{q}) = \sum_{t=1}^{4} \pi_t(\mathbf{pq})^2 X_t$$

$$\Pi_2(\mathbf{p},\mathbf{q}) = \sum_{t=1}^{4} \pi_t(\mathbf{pq})^2 Y_t$$

## The Prisoners' Dilemma

Eisert and Wilkins have analyzed the quantum version of the following Prisoners' Dilemma:

| | C | D |
|---|---|---|
| C | 3,3 | 0,5 |
| D | 5,0 | 1,1 |

GAME IX-4

This game has the following Nash equilibrium in mixed quantum strategies:

1. Player 1 randomizes over the quaternions 1 and $k$ with equal probability.
2. Player 2 randomizes over the quaternions $i$ and $j$ with equal probability.

This Nash equilibrium has an expected payoff of $\frac{5}{2}$

**Proof that the indicated pair of strategies is a Nash equilibrium:**

1. Fix player 1's strategy.
2. We want to show that player 2's strategy is optimal against player 1's strategy.
3. Suppose player 2 plays the quaternion strategy $\mathbf{q} = \alpha + \beta i + \gamma j + \delta k$

4. Then player 2's expected payoff equals:

$$
\begin{aligned}
\frac{1}{2}\Pi_2(1,\mathbf{q}) + \frac{1}{2}\Pi_2(k,\mathbf{q}) &= \frac{1}{2}\sum_{t=1}^{4}\pi_t(1\mathbf{q})^2 Y_t + \frac{1}{2}\sum_{t=1}^{4}\pi_t(k\mathbf{q})^2 Y_t \\
&= \frac{1}{2}\sum_{t=1}^{4}\left[\pi_t(\alpha+\beta i+\gamma j+\delta k)\right]^2 Y_t \\
&\quad + \frac{1}{2}\sum_{t=1}^{4}\left[\pi_t(-\delta+\gamma i+\beta j+\alpha k)\right]^2 Y_t \\
&= \frac{1}{2}(3\alpha^2+5\beta^2+\delta^2) \\
&\quad + \frac{1}{2}(3\delta^2+5\gamma^2+\alpha^2) \\
&= 2\alpha^2+\frac{5}{2}\beta^2+\frac{5}{2}\gamma^2+2\delta^2
\end{aligned}
$$

Hence an optimal response for player 2 solves the problem:

$$\begin{aligned} \max\ & 2\alpha^2 + \frac{5}{2}\beta^2 + \frac{5}{2}\gamma^2 + 2\delta^2 \\ s.t.\ & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1 \end{aligned}$$

Evidently, one solution to this problem is

$$\beta = \gamma = \frac{1}{2}.$$