Game Theory Principles IV

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Week 4: Game theory in political science

1: Voting, the median voter theorem.

Campaign promises in election: Should candidates try to appeal to all voters, or should they concentrate resources on subsets of the electorate? (inspired by Myerson, *American Political Science Review*, 1993)

Consider an environment in which

1. with a given electoral system, e.g. plurality voting,
2. initially all voters are the same,
3. candidates have identical fixed amounts of resources (campaign promises) to allocate
4. voters do not vote strategically.

Focus on the case of two candidates.

Refer to the campaign promises made by a candidate as her platform.

Observation: If platforms are chosen sequentially, the candidate who is the last to choose his platform always wins.

Reason: After observing the first candidate's platform, the second candidate can announce a platform that promises nothing to the voter who is promised the most by the first candidate, distribute the corresponding resources equally among the remaining candidates and otherwise replicate the first candidate's platform.

Therefore, we will focus on simultaneous platform choices.

A simplified game of platform choices:

1. there are two candidates
2. candidates compete for the votes of three voters
3. each voter votes for the candidate who promises her the most
4. each candidate has 9 units of promises to allocate
5. to keep the game within a manageable size we restrict the allocation of promises to a limited set of integer amounts: an equal allocation across voters, (3,3,3), and three allocations that only make promises to a strict subset of voters (5,4,0), (0,5,4) and (4,0,5).

6. A candidate wins the election and receives a payoff of 1 if she receives a plurality of the votes.
7. Voters who receive equal promises from both candidates are equally likely to vote for either candidate.
8. The losing candidate's payoff is -1.

Accordingly, we have the following payoff matrix (since we are dealing with a zero-sum game it suffices only to report the payoff of candidate 1):

Candidate 1 \ Candidate 2	(3,3,3)	(5,4,0)	(0,5,4)	(4,0,5)
(3,3,3)	0	-1	-1	-1
(5,4,0)	1	0	-1	1
(0,5,4)	1	1	0	-1
(4,0,5)	1	-1	1	0

Game IV-1

Observe that the strategy (3,3,3) is weakly dominated by any of the other pure strategies.

Furthermore, observe that the strategy (3,3,3) is strictly dominated by any non-degenerate mixture of the other strategies.

A strictly dominated strategy is never part of a Nash equilibrium. Hence, in searching for a Nash equilibrium we can concentrate on the strategies (5,4,0), (0,5,4) and (4,0,5).

One easily checks that there is no pure strategy equilibrium.

There is also no Nash equilibrium in which one player randomizes over only two of her strategies:

1. Suppose candidate 2 randomizes over (5,4,0) and (0,5,4).
2. Then candidate 1 never uses strategy (5,4,0).
3. But, then candidate 2 never uses strategy (0,5,4).
4. This implies that randomization over (5,4,0) and (0,5,4) by candidate 2 can never be part of a Nash equilibrium.

5. Following the same logic, one can show that there is no Nash equilibrium in which either candidate uses a non-degenerate randomization over only two of her pure strategies.

Hence, in any Nash equilibrium of our electoral competition game both candidates use a non-degenerate randomization over the strategies (5,4,0), (0,5,4) and (4,0,5).

By the indifference principle, in order for candidate 1 to randomize over her strategies the strategies (5,4,0), (0,5,4) and (4,0,5), she must be indifferent among these strategies.

Suppose candidate 2 uses the strategies (5,4,0), (0,5,4) and (4,0,5) with probabilities q_1, q_2 and q_3, respectively. Then

1. candidate 1's payoff from (5,4,0) equals

$$q_1 \times 0 + q_2 \times (-1) + q_3 \times 1$$

2. candidate 1's payoff from (0,5,4) equals

$$q_1 \times 1 + q_2 \times 0 + q_3 \times (-1)$$

3. candidate 1's payoff from (4,0,5) equals

$$q_1 \times (-1) + q_2 \times 1 + q_3 \times 0.$$

Candidate 1's indifference requires that all three of these expected payoffs are equal.

Since the q_i are probabilities, they have to sum to one.

Therefore, we get the following system of equations:

$$\begin{aligned} -q_2 + q_3 &= q_1 - q_3 \\ q_1 - q_3 &= -q_1 + q_2 \\ q_1 + q_2 + q_3 &= 1 \end{aligned}$$

Solve the last equation for q_1 to obtain

$$q_1 = 1 - q_2 - q_3$$

and substitute into both of the first two equations:

$$\begin{aligned} -q_2 + q_3 &= (1 - q_2 - q_3) - q_3 \\ (1 - q_2 - q_3) - q_3 &= -(1 - q_2 - q_3) + q_2 \end{aligned}$$

The first of these equations reduces to $q_3 = \frac{1}{3}$. Substituting this result into the lasts equation implies $q_2 = \frac{1}{3}$ and since probabilities have to sum to one also that $q_1 = \frac{1}{3}$.

We conclude that the column player's minmax strategy is the randomization $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ applied to the strategies (3,3,3), (5,4,0), (0,5,4) and (4,0,5).

By symmetry, the row player's maxmin strategy is identical (you should check the details of this derivation).

In Myerson's terms, in equilibrium there will be favored minorities.

Provision of a Public Good

1. Examples: Financing of a bridge, a public school system, concessions to reach a trade agreement etc.
2. We will focus on the two-agent case.
3. Both agents have the choice to either contribute or not to contribute to the provision of the public good.

4. Suppose each agent's preferred outcome is that the public good is provided and she does not contribute.
5. Each agent's second most preferred outcome is that the public good is provided and she contributes.
6. Each agent's least preferred outcome is that the public good is not provided.
7. Assumption: It suffices that one agent contributes in order for the public good to be provided.

The following game matrix presents a numerical example that is consistent with the agents' ordinal ranking of the various possible outcomes.

		Jill: contribute	Jill: not
Jack	contribute	(5,5)	(5,8)
	not	(8,5)	(1,1)

Game IV-2

Notice that both player agree that someone should contribute to the provision of the public good.

There is conflict over who should contribute.

Since there is common interest in one dimension and opposing interest in another dimension, this is a non-zero-sum game.

The mix of cooperative and conflicting motives in this game is expressed in the multiplicity of Nash equilibria in this game.

A natural question: How does the conflict over Nash equilibria get resolved in this game?

A few possible answers:

1. There may be a convention according to which Jack always contributes and Jill does not.
2. Jack and Jill may have a chance to talk before they make their decision. Without prejudging what the outcome of their conversation might be, evidently, if they reach an agreement, it must be one of the Nash equilibria of the game.
3. Repetition of the game might offer the opportunity to switch roles of contributor and non-contributor from period to period.

A more difficult question: What can we say about Jack and Jill's behavior if they have no opportunity to talk and no prior history to guide their actions?

A tentative answer: They are likely to be uncertain about each other's behavior.

A further question: Can we say more about the nature of this uncertainty?

As one step in the direction of an answer, suppose that both believe that the other is equally likely to contribute and not to contribute.

Notice that in this case both would want to contribute with probability one, which would contradict their beliefs.

A further question: Are there beliefs that express uncertainty and that are not contradicted by the behavior they induce?

A similar question: Are there mutually consistent beliefs?

If Jill is uncertain and believes that Jack will with positive probability contribute and with positive probability not contribute, and if Jack's behavior is consistent with Jill's beliefs, then Jack must be indifferent between both of his actions.

Genuine uncertainty coupled with mutually consistent beliefs corresponds to a mixed strategy equilibrium of the game.

If Jack believes that Jill contributes with probability q and that Jill does not contribute with probability $(1-q)$ then Jack's expected payoffs are:

$$\pi(\text{contribute}) = 5q + 5(1-q) = 5$$

$$\pi(\text{don't contribute}) = 8q + 1(1-q) = 1 + 7q$$

Therefore, for Jack to be indifferent, we need

$$\pi(\text{contribute}) = 5 = 1 + 7q = \pi(\text{don't contribute})$$

$$\Rightarrow q = \frac{4}{7}$$

By symmetry, for Jill to be indifferent, Jack also has to contribute with probability

$$p = \frac{4}{7}$$

Therefore, the game has a unique equilibrium in mixed strategies. This equilibrium is given by the probability pair

$$(p, q) = \left(\frac{4}{7}, \frac{4}{7}\right).$$

Observe that the mixed-strategy equilibrium of the public goods game is inefficient; there is positive probability that neither Jack nor Jill contributes, which is the least preferred outcome for both of them.

We can think of the inefficiency that is embodied in the mixed strategy equilibrium as expressing the possibility of a bargaining breakdown in a trade-negotiation, the possibility of a failure to provide a public good when too many potential contributors free-ride etc.

Similar Games - Different Stories

Rebel without a cause (1955):

1. Jim Stark's (James Dean) middle-class family is new in town.
2. They had to move to get Jim out of trouble.
3. Jim seeks respect and affection among his peers.
4. To prove himself, he participates in "chickie games," car races toward a seaside cliff.

Another **Chicken Game**

1. Jim and Buzz simultaneously drive toward the cliff.
2. Each has the choice to either jump out of the car or drive.
3. The winner is the one who jumps last.
4. Jumping later carries the risk that the car will go over the cliff.

		Buzz	
		jump	drive
Jim	jump	(5,5)	(2,8)
	drive	(8,2)	(-3,-3)

Game IV-3

Equilibria of Chicken:

1. Chicken has two pure-strategy equilibria: (jump, drive) and (drive, jump).
2. We will see that in addition the game has a unique mixed-strategy equilibrium (p^*, q^*), where p^* is the equilibrium probability that Jim jumps and q^* is the equilibrium probability that Buzz jumps.

Calculating the mixed-strategy Nash equilibrium:
Finding Buzz's equilibrium strategy q^*.

1. Buzz's strategy must make Jim indifferent.
2. For Jim to be indifferent, Jim's expected payoff from jump must equal his expected payoff from drive.
3. Jim's expected payoff from jump if Buzz jumps with probability q:
$$5q + 2(1 - q).$$
4. Jim's expected payoff from drive if Buzz jumps with probability q:
$$8q - 3(1 - q).$$

5. For Jim to be indifferent, his expected payoffs have to be equal (how's that for a tautology?), i.e.

$$5q + 2(1 - q) = 8q - 3(1 - q).$$

$$\Rightarrow 8q = 5$$
$$\Rightarrow q^* = \frac{5}{8}.$$

6. Since the game is symmetric, for Jim to make Buzz indifferent, Jim's strategy must satisfy:

$$p^* = \frac{5}{8}.$$

7. Hence, the unique mixed-strategy equilibrium of the game is

$$(p^*, q^*) = \left(\frac{5}{8}, \frac{5}{8}\right).$$

A general observation: In a mixed-strategy equilibrium of a two-player game, player one's strategy makes player two indifferent and vice versa.

Some Comparative Statics (Lots of Chicken games all at once):

1. Sometimes we want to know how social outcomes change as we vary the underlying conditions.
2. If we think of social outcomes as being determined by Nash equilibria, this is the same as asking: How do equilibria vary with parameters?

3. Suppose, we make Jim less averse to losing face by changing his payoff from (jump, drive) to $x > 2$ rather than 2, while leaving all the other payoffs in the game the same?

Jim \ Buzz	jump	drive
jump	(5,5)	(x,8)
drive	(8,2)	(-3,-3)

Game IV-4

In this class of games, the pure-strategy Nash equilibria do not change, as we increase x.

One could imagine that **breaking the symmetry among pure strategy-equilibria** might help resolve the uncertainty about which equilibrium is relevant.

If not, then we have to ask: How do the mixed-strategy equilibria change, as we increase x?

A first observation: Jim's mixing probability does not change, as we increase x.

Second observation: Buzz's mixing probability $q(x)$ is now a function of x and must satisfy

$$5q(x) + x(1 - q(x)) = 8q(x) - 3(1 - q(x))$$

in order to make Jim indifferent.

Solving the indifference condition:

$$5q(x) + x(1 - q(x)) = 8q(x) - 3(1 - q(x)) \Rightarrow$$

$$(6 + x)q(x) = 3 + x \Rightarrow$$

$$q(x) = \frac{3 + x}{6 + x}.$$

$q(x)$ is strictly increasing in x and converges to 1 as $x \longrightarrow \infty$.

Interpretation: As it becomes more attractive for Jim to jump, Buzz's mixing probability has to become more and more extreme to keep Jim from choosing to jump.

If we think it unlikely that Jim holds such extreme beliefs, raising x should eventually favor the pure strategy equilibrium $(jump, drive)$.

A related voting problem

1. Three voters vote on two alternatives.
2. One voter, voter 1, prefers alternative 1, the other two voters, voters 2 and 3, prefer alternative 2.
3. The payoff from seeing one's preferred alternative win is 2.
4. The payoff from seeing one's preferred alternative lose is 0.
5. The payoff from a tie is 1.
6. The cost of voting is $c > 0$ is small.

There is a public-goods aspect to voting: If voter 1 does not vote, then voter 2 would prefer voter 3 to vote and vice versa.

Some thoughts on pure strategy equilibria:
If only voter 2 votes . . . there is an incentive for voter 1 to vote . . . in which case there is an incentive for voter 3 to vote . . . in which case voter 1 prefers not to vote . . . in which case voter 2 prefers not to vote . . .

One easily verifies: There is no pure-strategy equilibrium.

A mixed strategy equilibrium:

Let us see whether there is a mixed strategy equilibrium in which voters 2 and 3 behave identically.

Let p denote the mixing probability of voter 1 and q the mixing probability of both voter 2 and voter 3.

Voter 1's payoff from voting:

$$q^2 \times 0 + 2q(1-q) \times 1 + (1-q)^2 \times 2 - c.$$

Voter 1's payoff from not voting:

$$q^2 \times 0 + 2q(1-q) \times 0 + (1-q)^2 \times 1.$$

Voter 1's indifference condition:

$$2q(1-q)+2(1-q)^2-c=(1-q)^2$$

$$\Rightarrow 2q(1-q)+(1-q)^2-c=0$$

$$\Rightarrow 2q-2q^2+1-2q+q^2-c=0$$

$$\Rightarrow q^2=1-c$$

$$\Rightarrow q=\sqrt{1-c}.$$

Voters 2 and 3 have the same indifference condition:

Voter 2's payoff from voting:

$$pq \times 2 + p(1-q) \times 1 + q(1-p) \times 2 + (1-p)(1-q) \times 2 - c$$

Voter 2's payoff from not voting:

$$pq \times 1 + p(1-q) \times 0 + q(1-p) \times 2 + (1-p)(1-q) \times 1$$

Voter 2's indifference condition implies that

$$p = 1 - \sqrt{1 - c}.$$