Game Theory Principles I

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INTRODUCTION: Examples of games, classroom experiments on *nim*, the *guessing game* and the *astronaut game*. Concepts: Agents, objectives, rules, strategies, conflict, cooperation, coordination. Tools: game trees, backward induction.

An enigmatic quote:

At this point the distinction between a low-marriage and a high-marriage society disappears. So these interacting agents, influenced by the choices one another makes, display the whole gamut of behaviours that characterize the particles of a fluid, influenced by their mutual forces of attraction and repulsion.

Philip Ball: "The Physics of Institutions"

We will conceive game theory broadly as the **theory of interactive behavior.**

Examples of Games

1. From economics: internet auctions
2. From biology: mate selection
3. From sports: bike racing
4. From politics: voting
5. From sociology: marriage behavior
6. From physics: interacting particles
7. From parlor games: poker

The three essential ingredients of a game:

1. Interacting entities
2. The rules of interaction
3. The forces that govern the interaction

We will refer to the three components of a game as

1. The **players** participating in the game.
2. The **sets of strategies** available to the players in the game.
3. The **payoffs** that players receive as a function of all players' strategy choices.

The concept of a **strategy** is central to game theory.

A strategy is a **fully described behavioral disposition.**

If, as we often will, we deal with purposeful players, a player's strategy is a **complete contingent plan.**

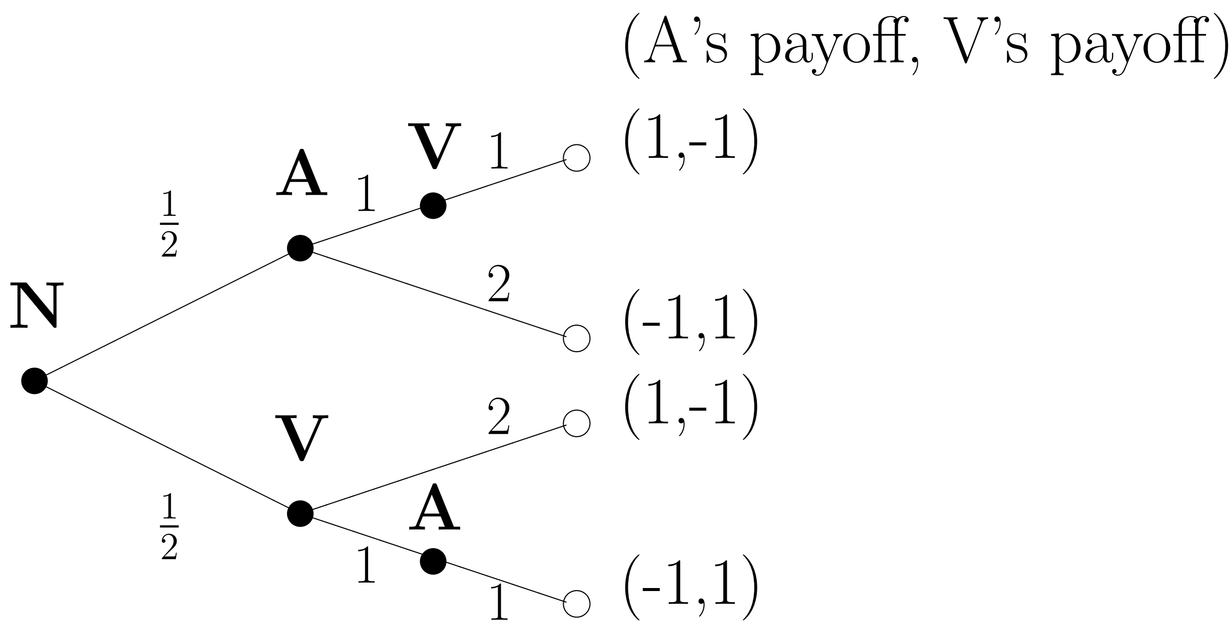
Let's play a simple game:

1. There are two players: Andreas and a volunteer.
2. There are ten tokens (indicated by chalk marks on the blackboard).
3. Andreas and the volunteer alternate taking tokens.
4. The player who starts taking tokens is determined by a coin toss.
5. When it is a player's turn, he is allowed to take either one or two tokens, no more no less.
6. The player who takes the last token loses.

A winning strategy for the second mover: "After each round in which the other player took two tokens, take one token. Otherwise, take two tokens."

Our goal is to find a **fool-proof method** for finding the solution of this game and of similar games.

An even simpler game, that starts with **two tokens:**



GAME I-1

This way of representing a game via a **game tree** is known as the **extensive form.**

A **game tree** consists of

1. An **initial** node (in our example the node marked with **N**).
2. **Decision nodes** (in our example indicated by solid circles).
3. **Terminal nodes** (in our example indicated by open circles).

4. An assignment of **players** (in our example **N**, **A** and **V**) to decision nodes.
5. **Branches** that start at decision nodes and which indicate the decisions available to the player acting at that node.
6. An assignment of **payoffs** to players at each terminal node.

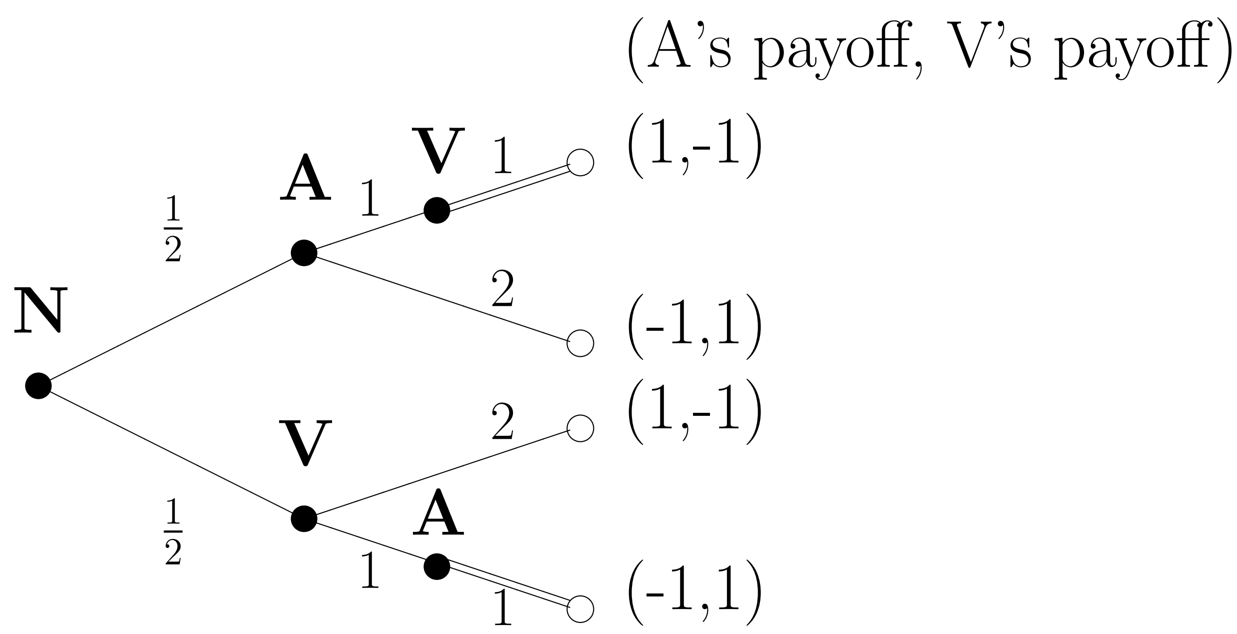
The player **N** in our game tree is a special player, called **Nature**.

Nature chooses according to exogenously fixed probabilities (in our example indicated by the **labels** "1/2" that are assigned to nature's two choices).

In our example, the remaining (genuine) choice nodes are labeled by the number of tokens a player takes with those choices.

Solving the Game

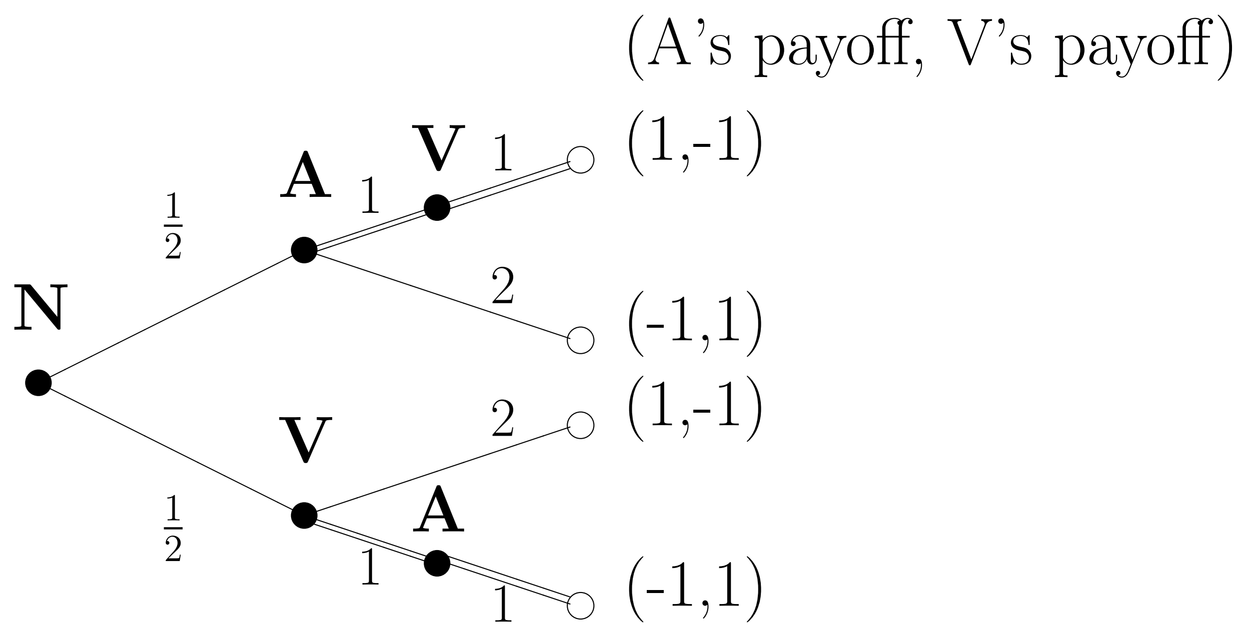
1. A player who moves with only one token left has no effective choice and loses the game.



GAME I-1

The inevitable choices are indicated by double lines.

2. A player who moves with two tokens left **anticipates** that if he leaves the other player with one token, the other player will lose the game.



GAME I-1

Two Solution Principles

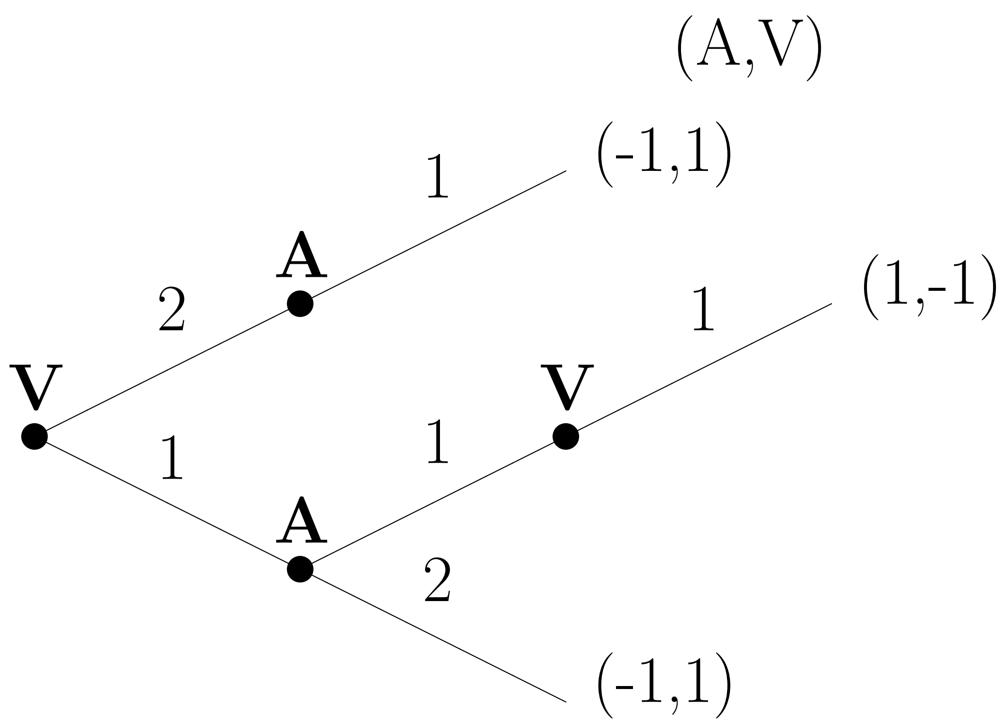
1. The principle that we applied in finding our solution is **anticipation.**
2. We will later identify an additional principle, **inference**, that will help us with solving more general games.

For now **anticipation** suffices:

1. For each node at which all choices lead to a terminal node, we can identify optimal choices.
2. Fix these choices.
3. For each node at which all choices either lead to a terminal node or to a node at which choices have been fixed, we can identify optimal choices.
4. Repeat the last step, until all nodes are associated with choices.

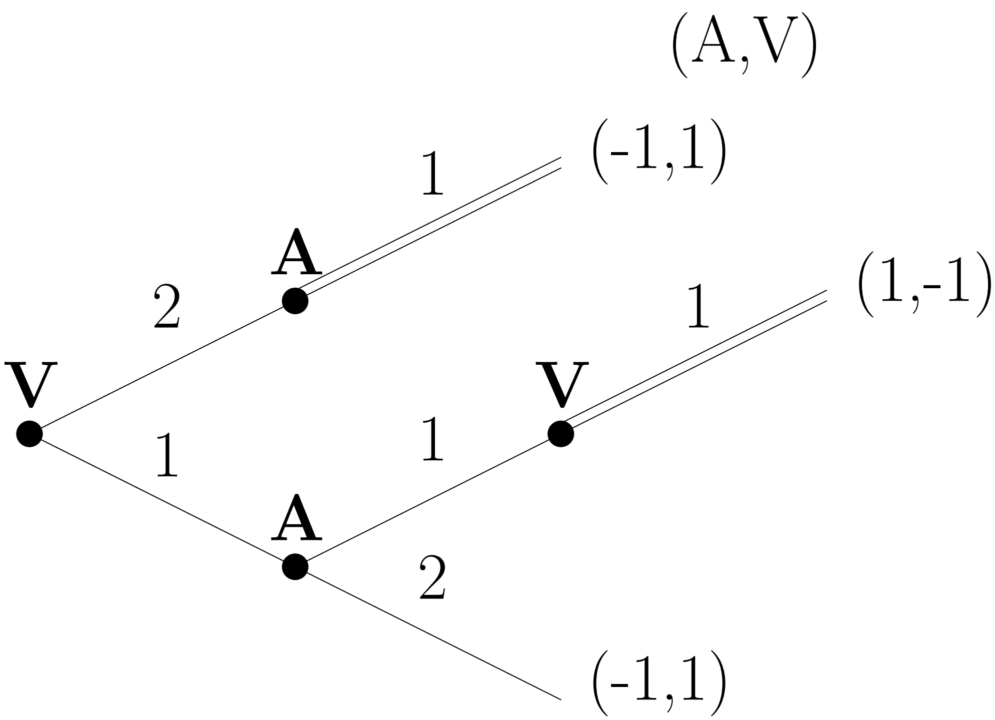
The application of the anticipation principle historically had been referred to as **backward induction**. This process of solving games backwards is sometimes also called **rollback**.

A slightly more complicated game, **Three tokens**, with the simplification that we eliminate nature.



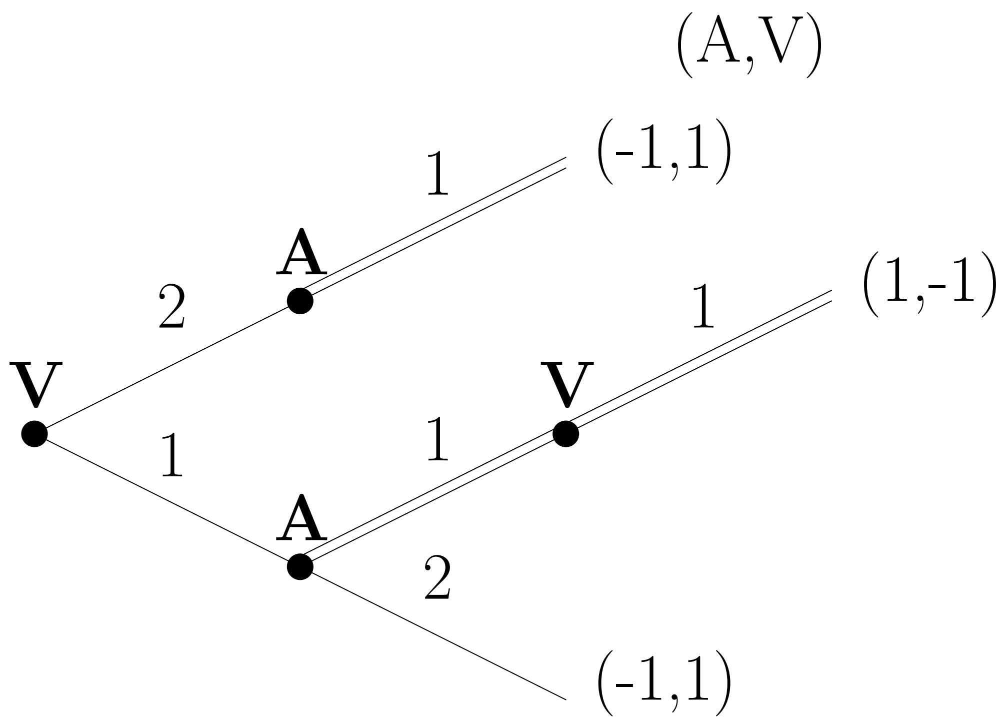
Game I-2

Step 1 of the solution of the three-token game:



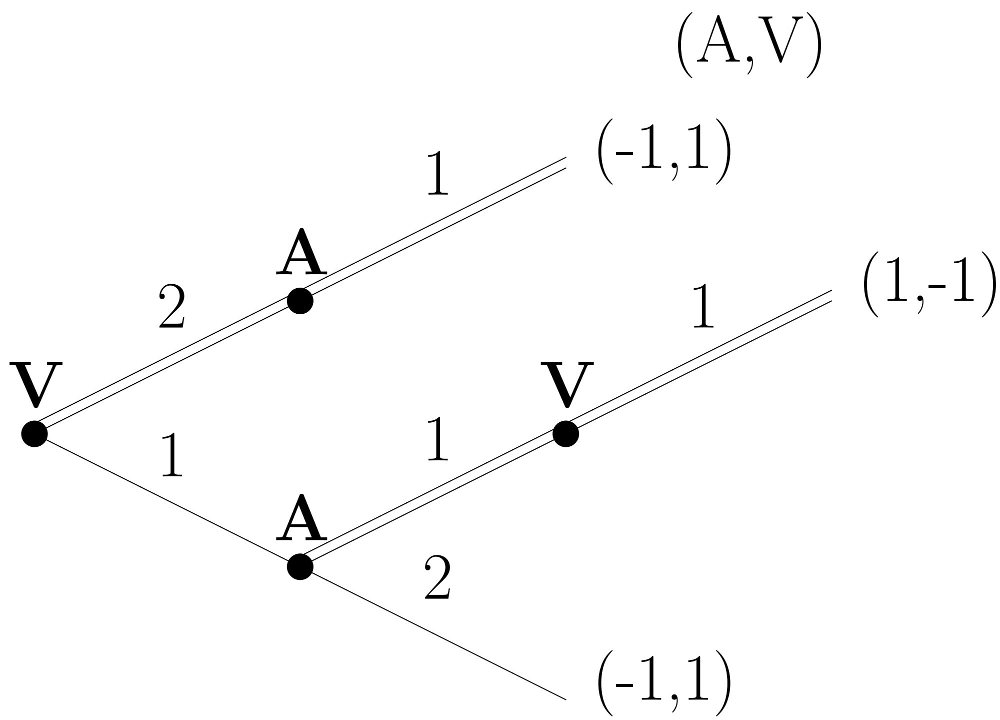
GAME I-2

Step 2 of the solution of the three-token game:



Game I-2

Step 3 of the solution of the three-token game:



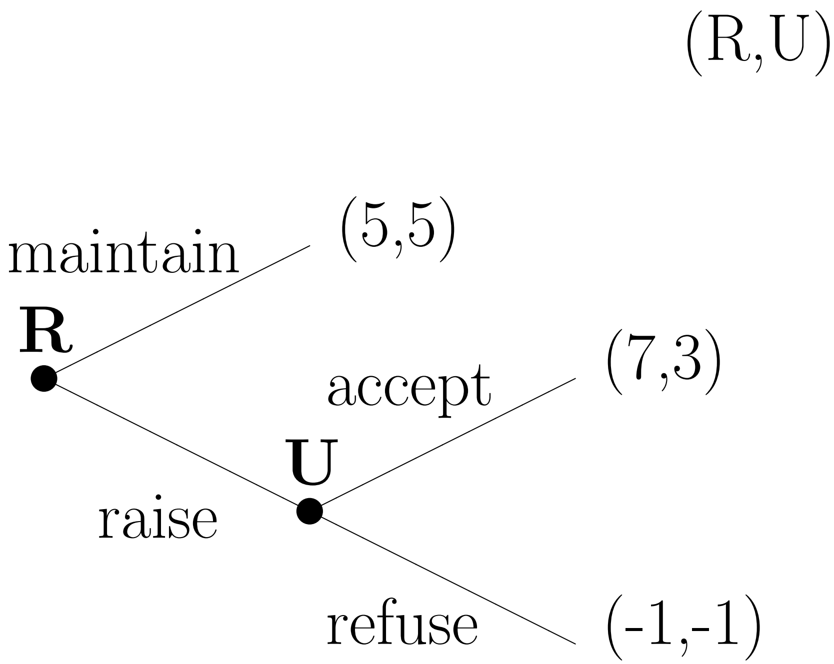
Game I-2

Once the solution has been obtained, the **outcome** that is induced by the solution can be found by following the **path** induced by the solution from the initial node.

A story from the real world:

1. On January 4, 2006 Russia and Ukraine resolved their disagreement over the price of natural gas.
2. Before, Russia had raised its price for gas shipments to Ukraine from a heavily subsidized level to the world market level.
3. Ukraine initially refused payment, which led to disruptions across Europe.
4. On January 4, 2006 accepted higher prices.
5. The new prices are still below world market prices.

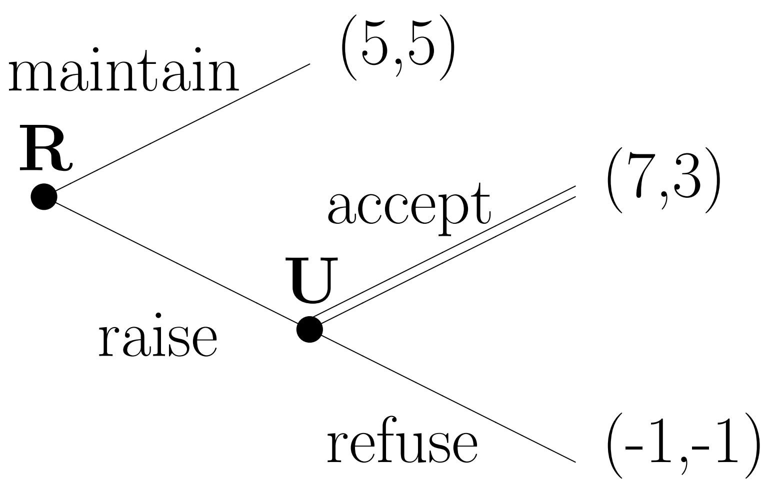
A stylized representation of the conflict between Russia, **R**, and the Ukraine, **U**:



GAME I-3

Solution of the Russia-Ukraine game (Step 1):

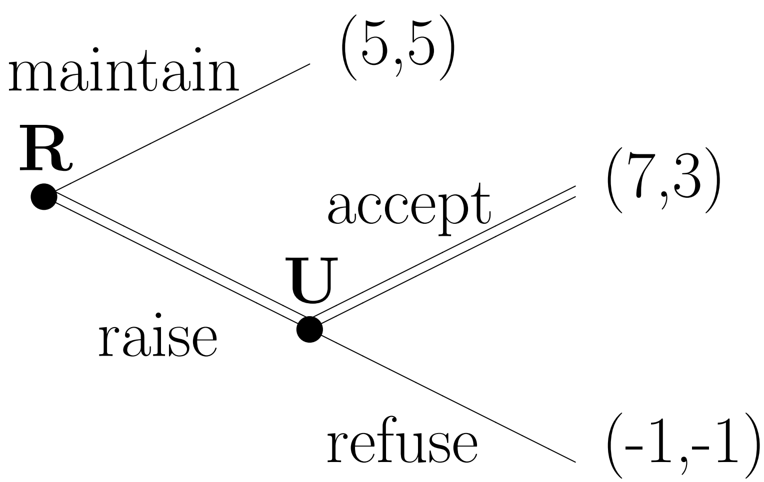
(R,U)



GAME I-3

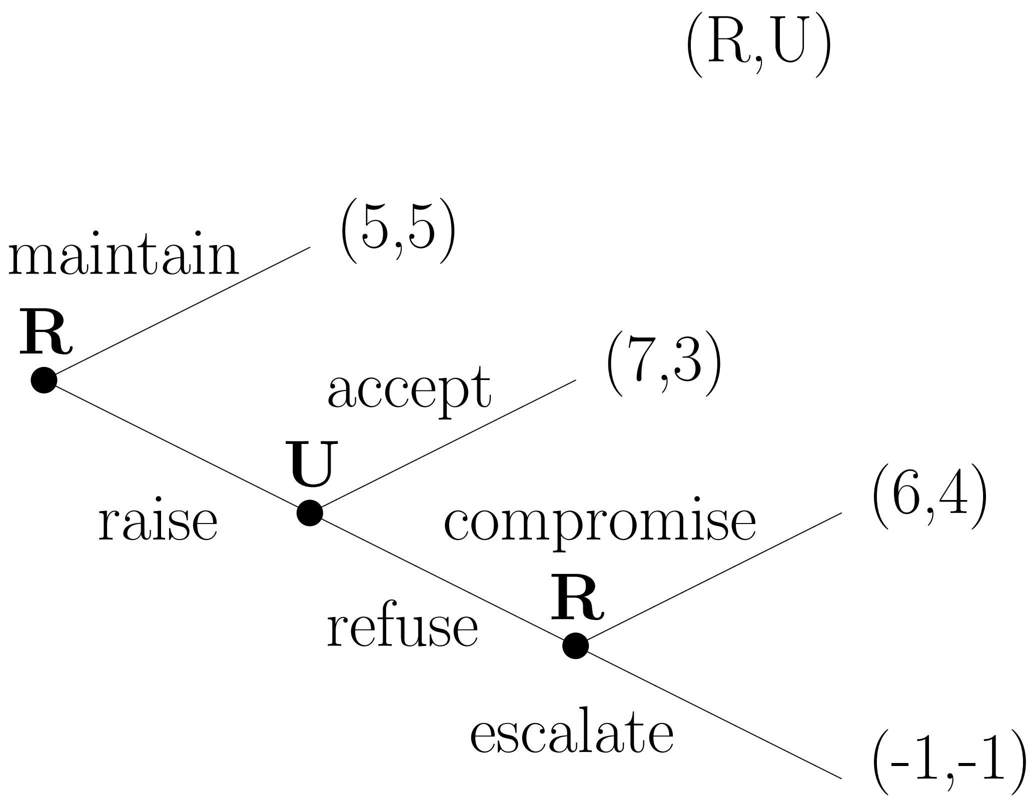
Solution of the Russia-Ukraine game (Step 2):

(R,U)



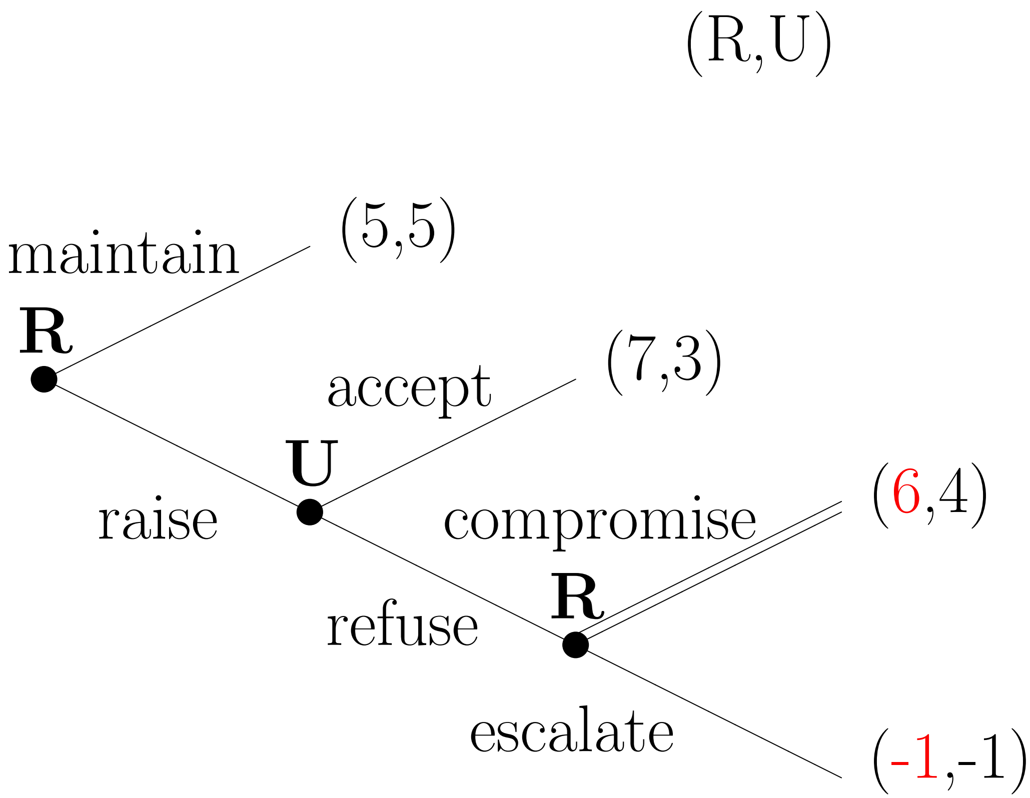
GAME I-3

It might be interesting to **enrich our description** of the Russia-Ukraine conflict. For example, what if a refusal by Ukraine were likely to elicit a more favorable offer by Russia? The game might look as follows:



Game I-4

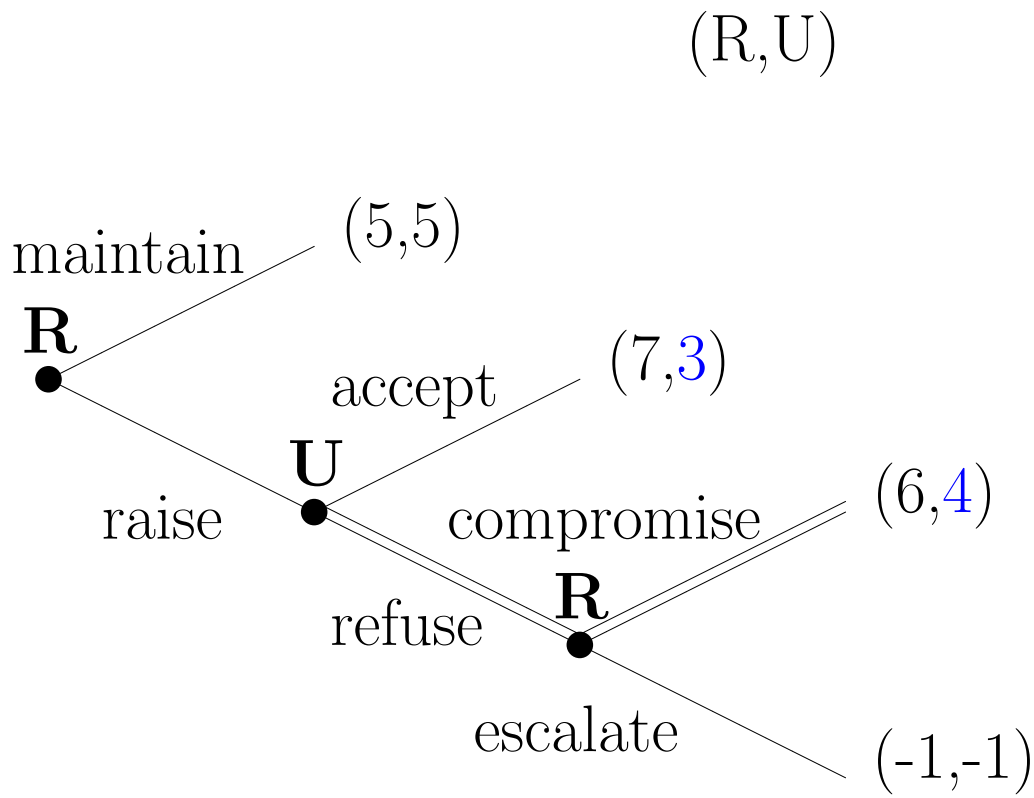
Solution of the enriched Russia-Ukraine game, using backward induction (Step 1):



GAME I-4

The payoffs that are relevant for this step have been indicated in red.

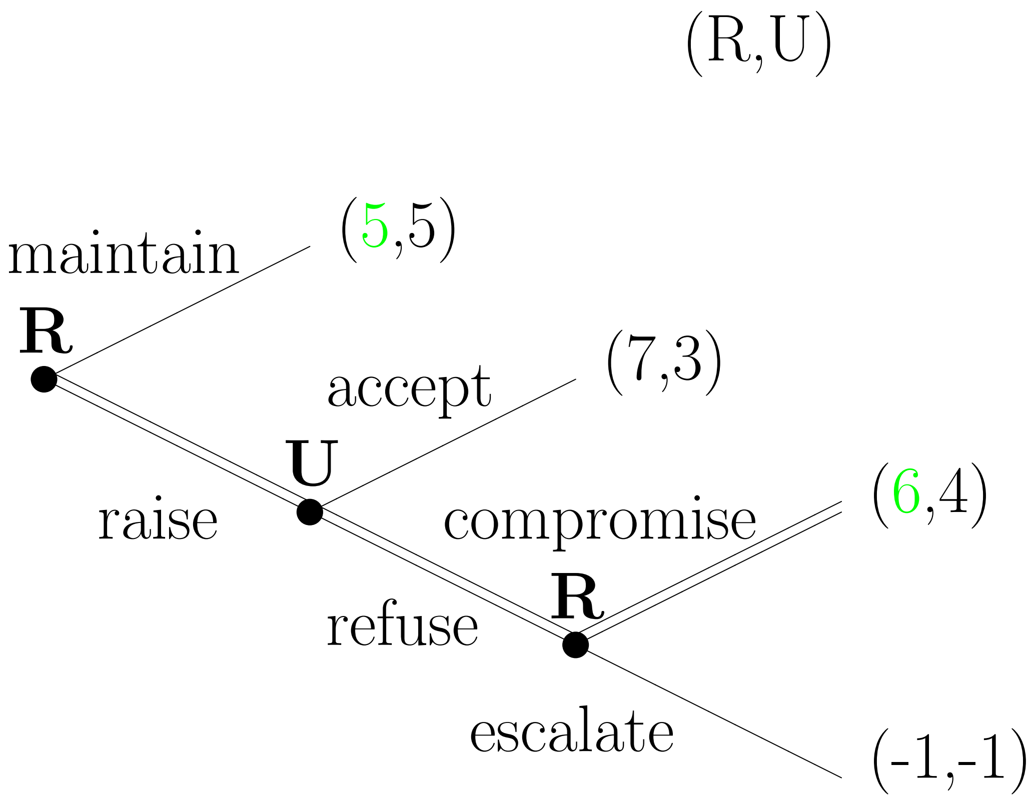
Solution of the enriched Russia-Ukraine game, using backward induction (Step 2):



Game I-4

The payoffs that are relevant for this step have been indicated in blue.

Solution of the enriched Russia-Ukraine game, using backward induction (Step 3, the final step):



Game I-4

The payoffs that are relevant for this step have been indicated in green.

A few words about strategies in Game I-4

1. Ukraine has only two **strategies in Game I-4**: "accept" and "refuse."
2. Russia has a more interesting set of strategies. For example, it can "raise" at its first node and "compromise" at its second node. We will represent this strategy by the pair **(raise, compromise)**. An alternative strategy is **(raise, escalate)**, i.e. to raise at its first node and to escalate once Ukraine refuses.
3. In total, Russia has for strategies: **(raise, compromise)**, **(raise, escalate)**, **(maintain, compromise)** and **(maintain, escalate)**.

Are there **redundant strategies** in our list?

Once Russia chooses to **maintain** its price, it does not matter how it would behave had it raised its price.

Therefore, can we ignore the distinction between **(maintain, compromise)** and **(maintain, escalate)**?

The answer is: no. A strategy will always completely specify behavior, even at contingencies that are ruled out by the strategy itself.

Why do we distinguish between strategies that induce identical behaviors?

1. The principal reason is that **players reason about each other:** We want all players to be able to reason through all the circumstances in which they may find themselves. Without a complete specification of Russia's strategy, Ukraine may be unable to evaluate its own strategies. Even if Ukraine expects "maintain," it wants to be prepared for the event "raise."
2. At some point, we will want to examine how **robust** strategies are to **error**. Then Russia cannot ignore the possibility that it chooses "raise" by mistake, even if it intended to choose "maintain."

Counting strategies:

Two-move chess:

1. In chess the first mover can choose among 20 possible actions.
2. The second mover, regardless of the first mover's actions choice, also has 20 available action choices.
3. Consider a version of chess that ends after each player has made one move. To make this a well-specified game, we need payoffs. For our purposes the exact specification of payoffs is irrelevant.

Question: How many strategies does the second mover have in two-move chess?

To answer this question it is worth recalling that a strategy is a *complete contingent plan.*

So, how many possible contingencies does the second mover face?

The answer is: all the possible moves of the first mover. That would be 20.

Now recall that for each of those 20 contingencies the second mover has 20 actions.

If we number the 20 actions available to the second mover from 1 to 20, an example of a **single strategy** for the second mover is

$$(7, 5, 5, 5, 7, 16, 18, 20, 3, 3, 3, 5, 5, 5, 11, 13, 17, 19, 1, 2)$$

This strategy prescribes to take action 7 following the first mover's action 1, to take action 5 following the first mover's action 2, to take action 5 following the first mover's action 3, and so on.

In this example, we can think of a strategy as a list with 20 entries. Each entry itself can take all values from 1 to 20.

There are $\underbrace{20 \times 20 \times \ldots \times 20}_{20 \text{ times}} = 20^{20}$ strategies.